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ON CALCULATION OF SOUND RAYS REFLECTED FROM NON-PLANE SURFACES IN ROOM ACOUSTICS

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In a three-dimensional case we develop a ray diffraction theory for arbitrary non-convex smooth obstacles with multiple ray re-reflections. Respective diffraction integrals are obtained from a certain generalization of the Kirchhoff physical diffraction theory. We present an asymptotic method to estimate the integrals on the basis of a stationary phase method. Explicit formulas for acoustic pressure in reflected wave are obtained, in the cases of single and double reflections.

1. Let us consider a classical problem about the scattering of a high-frequency wave, caused by a point-source, on arbitrary smooth surface *S*. The obstacle and the source are places in an acoustic medium. Dependence on the time is assumed to be monochromatic $p(x,y,z,t)=\mathrm{Re}[p(x,y,z)\exp(-i\omega t)]$, and the boundary *S* of the obstacle to be acoustically hard: $\partial p/\partial n|_S=0$. For a single reflection a solution of the problem in the two-dimensional case was obtained by various methods [1, 2]. In [2] there are obtained explicit asymptotic formulas for acoustic pressure in the reflected wave with arbitrary number of re-reflections, for the obstacle with a non-plane boundary. Here, for the study of a short-wave diffraction from obstacles of a complex shape we apply method [2], founded upon some estimate of the diffraction Kirchhoff integrals by a multi-dimensional stationary phase method. The present method has first allowed us to obtain an explicit result for the amplitude of a multiply reflected high-frequency acoustic wave.

2. Let from the point x_0 of acoustic medium a spherical high-frequency monochromatic wave fall to the surface *S* of the obstacle. In interaction between the wave and the obstacle, on the convex parts of the boundary surface *S* there are only points of simple specular reflections. On the concave parts of the surface *S* the diffraction field is more complex being defined by possible re-reflections of the incident wave [1]. Consideration of re-reflections on arrays of obstacles of complex shape is also very important. The proposed method is applicable in this case too. It is known that the pressure in the high-frequency wave at the point *x* is defined by an incident angle and a small vicinity of the point y^* of specular reflection from the surface *S*. Hence, with the frequency increasing the amplitude of the reflected signal can be obtained in frames of the ray theory on the basis of the stationary-phase method.

If a ray x_0 - y - x is reflected from the surface $S(y\in S)$ only once, then, according to the Kirchhoff physical diffraction theory, the pressure *p(x)* in the "light" zone is defined as follows [3]:

$$p(x)=\iint_S 2p^{inc}(y)\frac{\partial\Phi}{\partial n_y}dS\ , \tag{1}$$

where the boundary *S* of the obstacle is acoustically hard. Here $p^{inc}(y)$ - acoustical pressure in incident wave on the boundary *S*, Φ - a potential of fundamental solution (Green's function), n_y – external normal to the surface y of a small vicinity of *S* near the point y^* of specular reflection of the ray, *k* – wave number, γ - angle between the normal n_y and the direction of incidence of the ray x_0-y.

$$p^{inc}(y)=|x_0-y|^{-1}e^{ik|x_0-y|},\qquad \Phi=(4\pi)^{-1}|x-y|^{-1}e^{ik|x-y|} \tag{2}$$

and with $k\to\infty$

$$\frac{\partial\Phi}{\partial n_y}=ik\cos\gamma\,(4\pi)^{-1}|x-y|^{-1}e^{ik|x-y|}\left[1+O(k^{-1})\right] \tag{3}$$

Taking into account that both the incident $x_0 - y^*$ and reflected $y^* - x$ rays are on the same plane with a normal to the surface drawn in the point of specular reflection y^*, it can be obtained from (1) – (3) that the following main representation (after taking non-oscillating functions out of the integral, in the short-wave approximation) is valid:

$$p(x) = ik\frac{\cos\gamma}{L_0 L}\iint_S e^{ik\varphi}\,dS \tag{4}$$

$$\left|x_0 - y^*\right| = L_0,\ \left|y^* - x\right| = L,\ \varphi = \left|x_0 - y\right| + \left|y - x\right|, \tag{5}$$

Ray representation can be derived from (4), using the method of stationary phase [4]. Let us relate the surface *S* to a local Cartesian coordinate system, defined by the normal *n* and curvature lines of the surface at the point $y^* \in S$. Then arbitrary point $y \in S$ has the coordinates $y\left(\Delta s_1, \Delta s_2, -0.5\left(k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2\right)\right)$, where $\Delta s_1, \Delta s_2$ - small arcs along the curvature lines, $k_1 = R_1^{-1}$ and $k_2 = R_2^{-1}$ - principal curvatures, R_1 and R_2 – principal curvature radii at the point y^*, $k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2$ - the second quadratic form of the surface. Let us apply to the triangles $x_0 y^* y$ and xy^*y cosine theorem.

$$\left|x_0 - y\right|^2 = L_0^2 + \left|\mathbf{\Delta s}\right|^2 - 2L_0\left|\mathbf{\Delta s}\right|\cos\angle x_0 y^* y$$

$$\left|x - y\right|^2 = L^2 + \left|\mathbf{\Delta s}\right|^2 - 2L\left|\mathbf{\Delta s}\right|\cos\angle xy^* y \tag{6}$$

It follows from the scalar product of the vectors $\{\cos\alpha, \cos\beta, \cos\gamma\}$- ort of the vector $\boldsymbol{y^* x_0}$, $\mathbf{\Delta s} = \{\Delta s_1, \Delta s_2 - 0.5(k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2)\}$ and $\{-\cos\alpha, -\cos\beta, \cos\gamma\}$ - ort of the vector $\boldsymbol{y^* x}$, $\mathbf{\Delta s}$ that

$$\left|\mathbf{\Delta s}\right|\cos\angle x_0 y^* y = \Delta s_1\cos\alpha + \Delta s_2\cos\beta + 0.5\left(k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2\right)\cos\gamma$$

$$\left|\mathbf{\Delta s}\right|\cos\angle xy^* y = -\Delta s_1\cos\alpha - \Delta s_2\cos\beta + 0.5\left(k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2\right)\cos\gamma$$

If one neglects quantities, small with respect to $(\Delta s_1)^2, \Delta s_1\Delta s_2, (\Delta s_2)^2$, then it follows from eq.(6) that

$$\begin{aligned}\left|x_0 - y\right| = L_0 - \Delta s_1\cos\alpha - \Delta s_2\cos\beta + 0.5\left(L_0^{-1}\sin^2\alpha + k_1\cos\gamma\right)(\Delta s_1)^2 - \\ -L_0^{-1}\cos\alpha\cos\beta\,\Delta s_1\Delta s_2 + 0.5\left(L_0^{-1}\sin^2\beta + k_2\cos\gamma\right)(\Delta s_2)^2\end{aligned} \tag{7}$$

$$\begin{aligned}\left|x - y\right| = L + \Delta s_1\cos\alpha + \Delta s_2\cos\beta + 0.5\left(L^{-1}\sin^2\alpha + k_1\cos\gamma\right)(\Delta s_1)^2 - \\ -L^{-1}\cos\alpha\cos\beta\,\Delta s_1\Delta s_2 + 0.5\left(L^{-1}\sin^2\beta + k_2\cos\gamma\right)(\Delta s_2)^2\end{aligned} \tag{8}$$

Therefore, $\varphi = L_0 + L + 0.5d_{11}(\Delta s_1)^2 + d_{12}\Delta s_1\Delta s_2 + 0.5d_{22}(\Delta s_1)^2$

$$d_{11} = \left(L_0^{-1} + L^{-1}\right)\sin^2\alpha + 2k_1\cos\gamma;\ d_{12} = -\left(L_0^{-1} + L^{-1}\right)\cos\alpha\cos\beta; \tag{9}$$

$$d_{22} = \left(L_0^{-1} + L^{-1}\right)\sin^2\beta + 2k_2\cos\gamma$$

Absence of the first power of $\Delta s_1, \Delta s_2$ in the phase φ (5) proves, that the point y^* of direct ray reflection corresponds to a stationary value φ. The leading asymptotic term of the integral (4) at $k \to \infty$ is defined by coefficients before $(\Delta s_1)^2, \Delta s_1\Delta s_2, (\Delta s_2)^2$ and can be obtained from expression (4) by the two-dimensional stationary phase method [4].

$$p(x)=\frac{\exp\left\{i\left[k\ (L_0+L)+\frac{\pi}{4}(\delta_2+2)\right]\right\}}{\sqrt{L_0 L}\ \sqrt{|\det(D_2)|}}, \tag{10}$$

where D_2 - Gessian of symmetric structure $(d_{ij}=d_{ji};i,j=1,2)$ with elements (9). For more clear physical conclusions, in the case of single reflection, the obtained formula (10) can be also represented in more detail:

$$p(x)=\frac{\exp\left\{i\left[k(L_0+L_1)+\frac{\pi}{4}(\delta_2+2)\right]\right\}}{\sqrt{\left|(L_0+L)+2L_0L(L_0+L)\ (k_2\sin^2\alpha+k_1\sin^2\beta)\cos^{-1}\gamma+4L_0{}^2L^2K\right|}} \tag{11}$$

Here $K=k_1\,k_2$ – Gaussian curvature, $\boldsymbol{q}=\{-\cos\alpha,-\cos\beta,-\cos\gamma\}$ - vector, defining direction of the incident ray x_0-y^* in the chosen coordinate system, δ_2 - difference between the number of positive and negative eigenvalues of the Gessian matrix D_2.

Formula (11) is obtained for the case, when a high-frequency wave falls to a convex surface. If the wave is incident to a concave surface, the principal curvatures k_1 and k_2 should be taken negative. The structure of the obtained formula shows that, if the source and receiver are placed near the surface, then reflection of the wave is like from a tangential place drawn to the surface just at the point of specular reflection. With more distance from the source and receiver to the surface more important role is played by terms in the denominator, which relate to the shape of the surface through its principal curvatures. For more far distances the pressure in the reflected wave is completely defined by the Gaussian curvature, i.e. by the surface shape, and in a far-field approximation formula (11) coincides with a certain representation of [3].

3. For a double reflection of the ray $x_0-y_1^*-y_2^*-x_3$, radiated from the point-source x_0 and receiving in the point x_3, the received pressure $p(x_3)$ is given as follows:

$$p(x_3)=\iint_{S_2}2p(y_2)\frac{\partial\Phi}{\partial n_2}dS_2,\quad p(y_2)=\iint_{S_1}2p^{inc}(y_1)\frac{\partial\Phi}{\partial n_1}dS_1$$

Here $p(y_2)$ - the pressure in incident wave, which is defined after the first reflection from a small vicinity S_1. At the same time, the pressure $p(y_2)$ itself is expressed by similar formula. So, taking into account (2), one comes to the main representation:

$$p(x_3)=-\left(\frac{k}{4\pi}\right)^2\frac{\cos\gamma_1\cos\gamma_2}{L_0L_1L_2}\iint_{S_2}\iint_{S_1}e^{ik\varphi}dS_1dS_2 \tag{12}$$

$$\left|x_0-y_1^*\right|=L_0,\quad \left|y_1^*-y_2^*\right|=L_1,\quad \left|y_2^*-x_3\right|=L_2,\quad \varphi=|x_0-y_1|+|y_1-y_2|+|y_2-x_3| \tag{13}$$

Here y_1, y_2 - arbitrary surface points from small vicinities S_1 and S_2 of the points y_1^* и y_2^* of specular reflections. Let us relate the vicinities S_1 and S_2 of the specular reflection points y_1^*, y_2^* to right Cartesian coordinate systems, which are defined by the normals n_1 and n_2 to the curvature lines of the surface in the points y_1^*, y_2^*. The arc lengths $\Delta s_1{}^{(i)}$, $\Delta s_2{}^{(i)}$ are being counted from $y_i^*(i=1,2)$ along curvature lines. The first $|x_0-y_1|$ and the last $|y_2-x_3|$ terms in the phase φ (13) have the same structure as (7) and (8). The second term in(13) is:

$$
\begin{aligned}
|y_1 - y_2| &= L_1 + \Delta s_1^{(1)} \cos\alpha_1 + \Delta s_2^{(1)} \cos\beta_1 + 0.5\left(L_1^{-1}\sin^2\alpha_1 + k_1^{(1)}\cos\gamma_1\right) - \\
&- L_1^{-1}\cos\alpha_1\cos\beta_1 \Delta s_1^{(1)}\Delta s_2^{(1)} + 0.5\left(L_1^{-1}\sin^2\beta_1 + k_2^{(1)}\cos\gamma_1\right) - \\
&- \Delta s_1^{(2)}\cos\alpha_2 - \Delta s_2^{(2)}\cos\beta_2 - 0.5\left(L_1^{-1}\cos^2\alpha_2 - k_1^{(2)}\cos\gamma_2\right) - \\
&- L_1^{-1}\cos\alpha_2\cos\beta_2\Delta s_1^{(2)}\Delta s_2^{(2)} - 0.5\left(L_1^{-1}\cos^2\beta_2 - k_2^{(2)}\cos\gamma_2\right) + \\
&+ L_1^{-1}\sin^{-1}\gamma_1\sin^{-1}\gamma_2[(\cos\beta_1\cos\beta_2 - \cos\alpha_1\cos\alpha_2\cos\gamma_1\cos\gamma_2)\Delta s_1^{(1)}\Delta s_1^{(2)} - \\
&- (\cos\alpha_1\cos\beta_2 - \cos\beta_1\cos\alpha_2\cos\gamma_1\cos\gamma_2)\Delta s_2^{(1)}\Delta s_1^{(2)} - \\
&- (\cos\beta_1\cos\alpha_2 + \cos\alpha_1\cos\beta_2\cos\gamma_1\cos\gamma_2)\Delta s_1^{(1)}\Delta s_2^{(2)} + \\
&+ (\cos\alpha_1\cos\alpha_2 - \cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2)\Delta s_1^{(2)}\Delta s_2^{(2)}]
\end{aligned}
\tag{14}
$$

In this formula $k_1^{(i)}$, $k_2^{(i)}$ - principal curvatures of the surface at the points y_i^* $(i=1,2)$, $\{-\cos\alpha_1, -\cos\beta_1, -\cos\gamma_1\}$ - direction of the incident, and $\{-\cos\alpha_2, -\cos\beta_2, -\cos\gamma_2\}$ - of reflected rays, with respect to the coordinate system with the center y_2^*. It is clear from the structure of terms in the phase φ (13) that all first powers there $\Delta s_j^{(i)}(i,j=1,2)$ are absent. So, y_1^* and y_2^* correspond to a stationary value of the phase φ (13). $p(x_3)$ can be obtained from (12) by applying a four-dimensional stationary phase method [4].

$$
p(x_3) = \frac{\exp\left\{i\left[k\,(L_0 + L_1 + L_2) + \frac{\pi}{4}(\delta_4 + 4)\right]\right\}}{\sqrt{L_0 L_1 L_2}\,\sqrt{|\det\,(D_4)|}}, \tag{15}
$$

where $D_4 = (d_{ij})(i,j=1,2,3,4)$ - Gessian with the elements

$$
\begin{Bmatrix} d_{11} \\ d_{22} \end{Bmatrix} = 2\left[0.5\left(L_0^{-1} + L_1^{-1}\right)\begin{Bmatrix}\sin^2\alpha_1 \\ \sin^2\beta_1\end{Bmatrix} + \begin{Bmatrix}k_1^{(1)} \\ k_2^{(1)}\end{Bmatrix}\cos\gamma_1\right], \quad
\begin{Bmatrix} d_{33} \\ d_{44} \end{Bmatrix} = 2\left[0.5L_1^{-1} + 0.5\left(L_1^{-1} + L_2^{-1}\right)\begin{Bmatrix}\sin^2\alpha_2 \\ \sin^2\beta_2\end{Bmatrix} + \begin{Bmatrix}k_1^{(2)} \\ k_2^{(2)}\end{Bmatrix}\cos\gamma_2\right]
$$

$$
d_{12} = d_{21} = -\left(L_0^{-1} + L_1^{-1}\right)\cos\alpha_1\cos\beta_1, \qquad d_{34} = d_{43} = -\left(L_1^{-1} + L_2^{-1}\right)\cos\alpha_2\cos\beta_2
$$

$$
\begin{Bmatrix} d_{13} \\ d_{14} \end{Bmatrix} = \begin{Bmatrix} d_{31} \\ d_{41} \end{Bmatrix} = L_1^{-1}\sin^{-1}\gamma_1\sin^{-1}\gamma_2\left[\cos\beta_1\begin{Bmatrix}\cos\beta_2 \\ -\cos\alpha_2\end{Bmatrix} - \cos\alpha_1\begin{Bmatrix}\cos\alpha_2 \\ \cos\beta_2\end{Bmatrix}\cos\gamma_1\cos\gamma_2\right]
$$

$$
\begin{Bmatrix} d_{23} \\ d_{24} \end{Bmatrix} = \begin{Bmatrix} d_{32} \\ d_{42} \end{Bmatrix} = L_1^{-1}\sin^{-1}\gamma_1\sin^{-1}\gamma_2\left[\cos\alpha_1\begin{Bmatrix}-\cos\beta_2 \\ \cos\alpha_2\end{Bmatrix} + \cos\beta_1\begin{Bmatrix}\cos\alpha_2 \\ -\cos\beta_2\end{Bmatrix}\cos\gamma_1\cos\gamma_2\right],
$$

δ_4 - difference between numbers of positive and negative eigenvalues D_4.

Formulas (11) and (15), written in explicit form, show that the pressure *p(x)* in the reflected wave is defined by principal curvatures, by Gaussian curvature of the surface at the points of specular reflection, by distances from these points, by their distances from the source and receiver, and finally, by directions of incident and reflected waves. The proposed method will allow us to construct an explicit solution, in the ray approximation, for acoustic pressure with arbitrary number of re-reflections of high-frequency waves.

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